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Euclidean invariance and weak-equilibrium condition for the algebraic Reynolds stress model

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Taking into account the frame-invariance of a model expression under arbitrarily rotating transformations, Weis & Hutter (*J. Fluid Mech.* vol. 476, 2003, p. 63) proposed a Euclidean-objective weak-equilibrium condition for the algebraic Reynolds stress model (ARSM). However, Gatski & Wallin (*J. Fluid Mech.* vol. 518, 2004, p. 147) pointed out that the weak-equilibrium condition proposed is not correct in actual rotating flows such as a rotating channel flow and showed that a non-objective weak-equilibrium condition extended to curved and rotating flows should be assumed. The frame-invariance is an important issue not only for the ARSM but also for general nonlinear eddy-viscosity models. By introducing the corotational derivative of the Reynolds stress, the transport equation for the Reynolds stress can be written to be frame-invariant. It is shown that a frame-invariant expression is desirable as a general model by comparing the error of model expressions in different rotating frames. The extended weak-equilibrium condition of Gatski & Wallin is examined to show that it is in reality objective and it does not contradict a frame-invariant model expression for the Reynolds stress.

1. Introduction

In order to overcome the deficiencies associated with linear eddy-viscosity models, various nonlinear eddy-viscosity models have been developed (Yoshizawa 1984; Speziale 1987; Gatski & Jongen 2000). In particular, the explicit algebraic Reynolds stress model (ARSM) attracts interest because it represents a solution of the implicit ARSM that accurately treats the Reynolds stress anisotropy (Pope 1975; Gatski & Speziale 1993; Girimaji 1996; Wallin & Johansson 2000). In the ARSM, an algebraic equation is derived from the differential Reynolds stress model by assuming the weak-equilibrium condition; that is, the material derivative of the Reynolds stress anisotropy tensor is assumed to vanish. In the traditional ARSM for flows in a rotating frame the weak-equilibrium assumption is the same as that for an inertial frame. Weis & Hutter (2003) argued that existing models making use of this weak-equilibrium condition in a rotating frame are not frame-invariant under arbitrarily rotating (Euclidean) transformations and that a model expression should be frame-invariant because the choice of the coordinate system should not affect the adequacy of the model. To remedy this deficiency they proposed a Euclidean-objective weak-equilibrium condition to make models frame-invariant. However, Gatski & Wallin (2004) showed that the objective weak-equilibrium condition of Weis & Hutter (2003) is not correct in actual rotating flows such as rotating homogeneous shear and rotating channel flows. They stated that a non-objective weak-equilibrium condition taking into account flow rotation and curvature should be assumed and that a non-invariant model is justified because the transport equation for the Reynolds stress is not frame-invariant.

Euclidean invariance is an important property of physical laws. Whether a model expression for the Reynolds stress should be frame-invariant or not is an important issue not only for the ARSM but also for general nonlinear eddy-viscosity models. If the frame-invariance is required, it can be a useful constraint for theoretical modelling of nonlinear eddy-viscosity models. The time-derivative part of the transport equation for the Reynolds stress contains two additional terms involving the system rotation tensor. Gatski & Wallin (2004) argued that the transport equation is not frame-invariant because it involves the system rotation tensor. However, by introducing the corotational derivative of the Reynolds stress, the transport equation can be rewritten to be frame-invariant. It is expected that a model expression for the Reynolds stress should also be frame-invariant.

In this paper, we investigate the contradiction between the statement by Weis & Hutter (2003) that a model expression for the Reynolds stress should be frame-invariant and the objection by Gatski & Wallin (2004) that the objective weak-equilibrium condition is not correct. The fact that an objective condition proposed by Weis & Hutter (2003) is incorrect does not necessarily mean that weak-equilibrium conditions do not have to be objective. We will explain that the extended weak-equilibrium condition described by Gatski & Wallin (2004) is in reality objective and that the extended condition does not contradict a frame-invariant model expression for the Reynolds stress.

2. Objective variables and corotational derivative

Following Weis & Hutter (2003) we describe how the relevant variables appearing in the transport equation for the Reynolds stress transform between inertial and rotating frames. For simplicity, we consider the transformations between the two frames with the same origin. The space coordinates x_i^* in the inertial frame transform to the coordinates x_i in the rotating frame as

$$x_i = Q_{ij} x_j^*, \tag{2.1}$$

where Q_{ij} is an orthogonal transformation matrix. Variables expressed in the inertial frame are identified by asterisk. The system rotation tensor or the rotation rate of the x_i system expressed in the x_i system is given by

$$\varOmega_{ij} = Q_{ik} \frac{\mathrm{d}Q_{kj}^T}{\mathrm{d}t} = \varepsilon_{jik} \varOmega_k, \tag{2.2}$$

where Q_{ij}^T is the transpose of Q_{ij}, ε_{ijk} is the permutation tensor, and $\varOmega_i$ is the system rotation vector.

A vector f_i and a tensor f_{ij} that transform according to

$$f_i = Q_{ij} f_j^*, \quad f_{ij} = Q_{ik} f_{km}^* Q_{mj}^T, \tag{2.3}$$

are called objective variables. The transport equation for the Reynolds stress involves both objective and non-objective variables. The Reynolds stress $R_{ij}(=\langle u_i' u_j' \rangle)$, the Reynolds stress anisotropy tensor $a_{ij} = R_{ij}/K - \frac{2}{3}\delta_{ij}$, and the mean strain-rate tensor $S_{ij}[=(\partial U_i/\partial x_j + \partial U_j/\partial x_i)/2]$ are objective, where $U_i(=\langle u_i \rangle)$ and $u_i'(=u_i - U_i)$ represent the mean and fluctuating velocity fields and $K(=\langle u_i'^2 \rangle/2)$ is the turbulent kinetic energy. On the other hand, the mean vorticity tensor $W_{ij}\,[=(\partial U_i/\partial x_j - \partial U_j/\partial x_i)/2]$ is not objective because it transforms as

$$W^*_{ij} = Q^T_{ik}(W_{km} + \Omega_{km})Q_{mj}. \tag{2.4}$$

The mean vorticity tensor can be made objective by adding the system rotation tensor:

$$\bar{W}_{ij} = W_{ij} + \Omega_{ij}, \tag{2.5}$$

where $\bar{W}_{ij}$ is called the mean absolute vorticity tensor.

Similarly, the material derivative of the Reynolds stress, $\mathrm{D}R_{ij}/\mathrm{D}t$ (where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + U_i\partial/\partial x_i$), is not objective because it transforms as

$$\frac{\mathrm{D}R^*_{ij}}{\mathrm{D}t}\left(\equiv \frac{\partial R^*_{ij}}{\partial t} + U^*_k\frac{\partial R^*_{ij}}{\partial x^*_k}\right) = Q^T_{ik}\left(\frac{\mathrm{D}R_{km}}{\mathrm{D}t} + \Omega_{kn}R_{nm} - R_{kn}\Omega_{nm}\right)Q_{mj}. \tag{2.6}$$

Like the mean absolute vorticity tensor $\bar{W}_{ij}$, this material derivative can be made objective by adding terms involving the system rotation tensor:

$$\frac{\bar{\mathrm{D}}R_{ij}}{\mathrm{D}t} = \frac{\mathrm{D}R_{ij}}{\mathrm{D}t} + \Omega_{ik}R_{kj} - R_{ik}\Omega_{kj}. \tag{2.7}$$

Here, we call $\bar{\mathrm{D}}R_{ij}/\mathrm{D}t$ the corotational derivative of R_{ij} (Thiffeault 2001). The material derivative of the anisotropy tensor a_{ij} also transforms like (2.6) and its corotational derivative $\bar{\mathrm{D}}a_{ij}/\mathrm{D}t$ can be defined in the same form as (2.7).

Next, we examine the transformation properties of the transport equation for the Reynolds stress. The transport equation for R_{ij} in a rotating frame can be written as

$$\frac{\mathrm{D}R_{ij}}{\mathrm{D}t} + \Omega_{ik}R_{kj} - R_{ik}\Omega_{kj} = -R_{ik}(S_{kj} - \bar{W}_{kj}) - (S_{ik} + \bar{W}_{ik})R_{kj} - \varepsilon_{ij} + \Pi_{ij} + D_{ij}, \tag{2.8}$$

where ε_{ij}, Π_{ij}, and D_{ij} are the dissipation, pressure–strain, and diffusion terms, respectively. The three terms can be considered objective because they are expressed in terms of objective variables such as the velocity and pressure fluctuations and their spatial derivatives (detailed expressions are omitted here). For the transport equation for a_{ij} corresponding to (2.8), Gatski & Wallin (2004) noted that the complete left-hand side represents the advection of the anisotropy tensor in a rotating frame and the right-hand side is written in terms of objective tensors, but the transport equation is not frame-invariant because it explicitly contains the system rotation tensor Ω_{ij}. However, as shown in Weis & Hutter (2003), (2.8) can be rewritten using the corotational derivative of R_{ij} as follows:

$$\frac{\bar{\mathrm{D}}R_{ij}}{\mathrm{D}t} = -R_{ik}(S_{kj} - \bar{W}_{kj}) - (S_{ik} + \bar{W}_{ik})R_{kj} - \varepsilon_{ij} + \Pi_{ij} + D_{ij}. \tag{2.9}$$

Since the corotational derivative $\bar{\mathrm{D}}R_{ij}/\mathrm{D}t$ can be considered as an objective variable like $\bar{W}_{ij}$, we believe that (2.9) is frame-invariant and hence a model expression for the Reynolds stress should also be frame-invariant as stated by Weis & Hutter (2003).

3. Weak-equilibrium condition for the algebraic Reynolds stress model

Now, we investigate the transformation properties of the ARSM equation. To obtain an explicit model expression, a quasi-linear model for the pressure–strain and dissipation terms $\Pi_{ij} - \varepsilon_{ij}$ is considered (Wallin & Johansson 2002). The transport equation for the anisotropy tensor a_{ij} in a rotating frame is then written as

$$\frac{\tau}{A_0}\left(\frac{\bar{\mathrm{D}}a_{ij}}{\mathrm{D}t} - D_{ij}^{(a)}\right) = \left(A_3 + A_4\frac{P}{\varepsilon}\right)a_{ij} + A_1\hat{S}_{ij} - (a_{ik}\hat{\bar{W}}_{kj} - \hat{\bar{W}}_{ik}a_{kj}) + A_2\left(a_{ik}\hat{S}_{kj} + \hat{S}_{ik}a_{kj} - \tfrac{2}{3}a_{km}\hat{S}_{mk}\delta_{ij}\right), \tag{3.1}$$

where

$$\tau = \frac{K}{\varepsilon}, \quad \hat{S}_{ij} = \tau S_{ij}, \quad \hat{\bar{W}}_{ij} = \tau \bar{W}_{ij}, \tag{3.2}$$

$$D_{ij}^{(a)} = \frac{1}{K}\left(D_{ij} - \frac{R_{ij}}{2K}D_{kk}\right), \quad P = -R_{ij}S_{ij}, \tag{3.3}$$

and ε is the turbulent energy dissipation rate and A_0–A_4 are model constants.

The transport equation for the anisotropy tensor in an inertial frame is obtained from (3.1) by replacing $\bar{\mathrm{D}}a_{ij}/\mathrm{D}t$ and $\hat{\bar{W}}_{ij}$ by $\mathrm{D}a_{ij}/\mathrm{D}t$ and $\hat{W}_{ij}$, respectively. An algebraic equation for a_{ij}^* in the inertial frame can be derived by assuming the weak-equilibrium condition

$$\frac{\mathrm{D}a_{ij}^*}{\mathrm{D}t} = 0, \tag{3.4}$$

and by assuming the following condition for the diffusion term:

$$D_{ij}^{(a)*} = 0. \tag{3.5}$$

The resulting equation is the implicit ARSM that accurately treats the Reynolds stress anisotropy. Its explicit solution can be obtained from linear algebra using integrity bases (Pope 1975; Gatski & Speziale 1993). The anisotropy tensor is then expressed explicitly in terms of the mean strain-rate and vorticity tensors as

$$a_{ij}^* = f_{ij}(\hat{S}_{km}^*, \hat{W}_{km}^*), \tag{3.6}$$

where detailed expressions for f_{ij} are given in Wallin & Johansson (2000).

Similarly, in the traditional ARSM for flows in a rotating frame, the conditions

$$\frac{\mathrm{D}a_{ij}}{\mathrm{D}t} = 0, \quad D_{ij}^{(a)} = 0, \tag{3.7}$$

are assumed in (3.1). The left-hand side of (3.1) is now rewritten as

$$\frac{\tau}{A_0}\frac{\bar{\mathrm{D}}a_{ij}}{\mathrm{D}t} = -\frac{\tau}{A_0}(a_{ik}\Omega_{kj} - \Omega_{ik}a_{kj}). \tag{3.8}$$

Since the right-hand side of (3.8) can be incorporated into the third term on the right-hand side of (3.1) involving $\hat{\bar{W}}_{ij}$, the solution of the ARSM for flows in a rotating frame can be given by

$$a_{ij} = f_{ij}\left(\hat{S}_{km}, \hat{\bar{W}}_{km} - \frac{\tau}{A_0}\Omega_{km}\right), \tag{3.9}$$

where f_{ij} is the same function as in (3.6) for the inertial frame. Since (3.9) involves the system rotation tensor Ω_{ij}, it is not frame-invariant. The non-invariance is because the weak-equilibrium condition $\mathrm{D}a_{ij}/\mathrm{D}t = 0$ is not objective.

To remove this artifact and to make the model frame-invariant Weis & Hutter (2003) proposed an objective weak-equilibrium condition

$$\frac{\bar{\mathrm{D}}a_{ij}}{\mathrm{D}t} = 0, \tag{3.10}$$

instead of $\mathrm{D}a_{ij}/\mathrm{D}t=0$. Since the left-hand side of (3.1) vanishes, the solution of the ARSM for flows in a rotating frame can then be written as

$$a_{ij} = f_{ij}(\hat{S}_{km}, \hat{\hat{W}}_{km}), \tag{3.11}$$

using the same function f_{ij} as in (3.6). Equation (3.11) is frame-invariant because it is expressed in terms of objective tensors only. However, Gatski & Wallin (2004) pointed out that the condition (3.10) is not correct in actual rotating flows. For example, the condition $\mathrm{D}a_{ij}/\mathrm{D}t=0$ exactly holds and $\bar{\mathrm{D}}a_{ij}/\mathrm{D}t=0$ is not satisfied in a rotating channel flow relative to the observer rotating with the channel. They suggested that a proper weak-equilibrium condition should be

$$\frac{\mathrm{D}a_{ij}^{\dagger}}{\mathrm{D}t}\left(\equiv \frac{\partial a_{ij}^{\dagger}}{\partial t} + U_k^{\dagger}\frac{\partial a_{ij}^{\dagger}}{\partial x_k^{\dagger}}\right) = 0, \tag{3.12}$$

where $a_{ij}^{\dagger}$ is the anisotropy tensor expressed in an appropriate frame representing flow rotation and curvature (it will be discussed in detail in the next section). In the case of a rotating channel flow, $a_{ij}^{\dagger}$ equals a_{ij}. They argued that the resulting model expression is not frame-invariant and that this is justified because the transport equation for the Reynolds stress is not frame-invariant.

As discussed in the preceding section, we believe that a model expression for the Reynolds stress should be frame-invariant. The fact that an objective condition $\bar{\mathrm{D}}a_{ij}/\mathrm{D}t=0$ proposed by Weis & Hutter (2003) is incorrect does not necessarily mean that weak-equilibrium conditions do not have to be objective. We expect that a proper objective condition can be found. Before seeking such a condition, we explain the reason why a frame-invariant expression is desirable by comparing the error of invariant and non-invariant model expressions as follows.

We consider two rotating frames A and B whose coordinates $x_i^{(A)}$ and $x_i^{(B)}$ transform into each other as

$$x_i^{(B)} = P_{ij}x_j^{(A)}, \tag{3.13}$$

where P_{ij} is an orthogonal transformation matrix. We examine two types of model expression:

$$a_{ij} = g_{ij}(\hat{S}_{km}, \hat{\hat{W}}_{km}), \tag{3.14}$$

and

$$a_{ij} = g_{ij}(\hat{S}_{km}, \hat{\hat{W}}_{km} + C\tau\Omega_{km}), \quad C \neq 0, \tag{3.15}$$

where g_{ij} and C are a non-dimensional function and constant, respectively. The model expressions can contain other objective tensors, but the above expressions are examined for simplicity. The first-type of expression (3.14) is frame-invariant whereas the second, (3.15), is not frame-invariant owing to Ω_{km}, like (3.9). For both types, we examine the error of a model expression defined as

$$E = \left(a_{ij} - a_{ij}^{true}\right)^2, \tag{3.16}$$

where a_{ij}^{true} is the true value of the anisotropy tensor. For the first model, the error $E^{(B)}$ in the rotating frame B is equal to $E^{(A)}$ in the rotating frame A as follows:

$$\begin{aligned} E^{(B)} &= \left(\boldsymbol{g}\left(\hat{\boldsymbol{S}}^{(B)}, \hat{\hat{\boldsymbol{W}}}^{(B)}\right) - \boldsymbol{a}^{true(B)}\right)^2 = \left(\boldsymbol{P}\boldsymbol{g}\left(\hat{\boldsymbol{S}}^{(A)}, \hat{\hat{\boldsymbol{W}}}^{(A)}\right)\boldsymbol{P}^T - \boldsymbol{P}\boldsymbol{a}^{true(A)}\boldsymbol{P}^T\right)^2 \\ &= \left(\boldsymbol{g}\left(\hat{\boldsymbol{S}}^{(A)}, \hat{\hat{\boldsymbol{W}}}^{(A)}\right) - \boldsymbol{a}^{true(A)}\right)^2 = E^{(A)}. \end{aligned} \tag{3.17}$$

Here, matrix notation is used to simplify the form of the equation. On the other hand, for the second model, the error is different in the two rotating frames as follows:

$$\begin{aligned} E^{(B)} &= \left(\boldsymbol{g}\left(\hat{\boldsymbol{S}}^{(B)}, \hat{\hat{\boldsymbol{W}}}^{(B)} + C\tau\boldsymbol{\Omega}^{(B)}\right) - \boldsymbol{a}^{true(B)}\right)^2 \\ &= \left(\boldsymbol{P}\boldsymbol{g}\left(\hat{\boldsymbol{S}}^{(A)}, \hat{\hat{\boldsymbol{W}}}^{(A)} + C\tau\boldsymbol{P}^T\boldsymbol{\Omega}^{(B)}\boldsymbol{P}\right)\boldsymbol{P}^T - \boldsymbol{P}\boldsymbol{a}^{true(A)}\boldsymbol{P}^T\right)^2 \\ &= \left(\boldsymbol{g}\left(\hat{\boldsymbol{S}}^{(A)}, \hat{\hat{\boldsymbol{W}}}^{(A)} + C\tau\boldsymbol{P}^T\boldsymbol{\Omega}^{(B)}\boldsymbol{P}\right) - \boldsymbol{a}^{true(A)}\right)^2 \neq E^{(A)}, \end{aligned} \tag{3.18}$$

because

$$\Omega_{ij}^{(A)} - P_{ik}^T\Omega_{km}^{(B)}P_{mj} = P_{ik}^T\frac{\mathrm{d}P_{kj}}{\mathrm{d}t} \neq 0. \tag{3.19}$$

These results do not necessarily preclude the second model. To simulate a rotating flow that has a trivial proper rotating frame, the second model that has a minimal error in this rotating frame can be used. In fact, the ARSM for a rotating channel flow using the condition $\mathrm{D}a_{ij}/\mathrm{D}t = 0$ in the frame rotating with the channel is appropriate because this condition exactly holds. However, the second model is not adequate as a model for more general flows for which a proper rotating frame is not trivial. For example, a flow in an annulus between inner and outer cylinders rotating about their common axis at different rotation rates can be simulated in a frame rotating with either the inner cylinder or the outer one.

To demonstrate the dependence of the second model on the frame of the observer, we do an *a priori* test of the models using results of the direct numerical simulation (DNS) of a turbulent flow in a concentric annulus with inner wall rotation (Okamoto & Shima 2005). The flow is calculated in the cylindrical coordinate system (r,θ,z) and variables are non-dimensionalized by the radial half-width δ and the axial global friction velocity $u_{\tau g} = \sqrt{-\delta \mathrm{d}P/\mathrm{d}z}$. The Reynolds number is set to $Re(\equiv \delta u_{\tau g}/\nu) = 150$. The radii of the inner and outer walls are given by $r_{in} = 2$ and $r_{out} = 4$, respectively, and the rate of the inner wall rotation is set to $\Omega_{z0} = 5$. Applying DNS data for U_i, k, and ε to model expressions, we evaluate the anisotropy tensor and compare results with the exact value. Figure 1 shows profiles of the anisotropy $a_{r\theta}$ obtained from (3.14) and (3.15) as a function of r. The explicit ARSM of Wallin & Johansson (2000) is used for function g_{ij} and the constant in (3.15) is set to $C = 9/4$. Three values of the system rotation rate are adopted: $\Omega_z = 5$ and 0 correspond to the systems rotating with the inner and outer walls, respectively, and $\Omega_z = 1.1$ represents the same rotation rate as the mean motion of a fluid at $r = 3$. Since (3.14) is frame-invariant, the profiles of $a_{r\theta}$ for (3.14) in the three systems are the same. On the other hand, the profiles obtained from (3.15) in the three systems are quite different; the value is underpredicted for $2.5 < r < 3.5$ in the case of $\Omega_z = 5$. Even the profile for (3.14) deviates from the exact value for $2.1 < r < 2.7$. The model needs to be improved in future work; here we concentrate on the dependence of the model on the reference frame. We evaluate the error of each model using the expression

$$E_{r\theta} = \frac{2}{r_{out}^2 - r_{in}^2}\int_{r_{in}}^{r_{out}} r\left(\frac{a_{r\theta} - a_{r\theta}^{true}}{a_{r\theta}^{true}}\right)^2 \mathrm{d}r. \tag{3.20}$$

For the first model (3.14), $E_{r\theta} = 0.089$ is obtained in the three systems. In the case of the second model (3.15), $E_{r\theta} = 0.089$ for $\Omega_z = 0$, $E_{r\theta} = 0.11$ for $\Omega_z = 1.1$, and $E_{r\theta} = 0.79$ for $\Omega_z = 5$. Therefore, the second model predicts results with different accuracy depending on the choice of the frame of the observer. On the other hand, the error of the first model does not depend on the frame; simulation results obtained

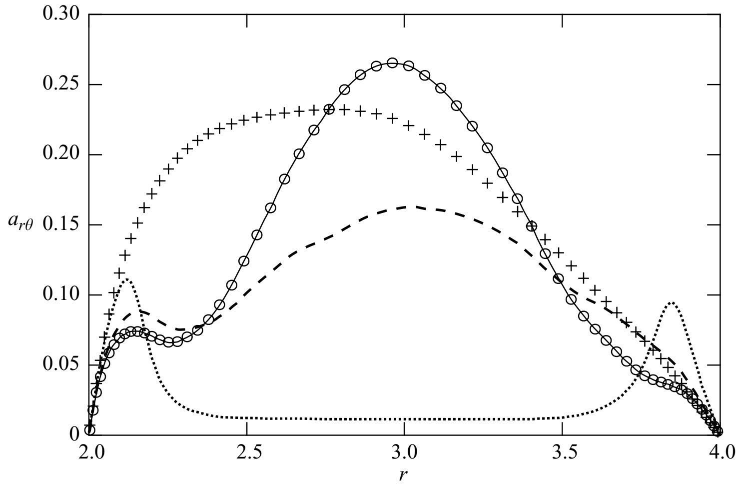

FIGURE 1. Profiles of anisotropy $a_{r\theta}$ of a turbulent flow in a concentric annulus with inner cylinder rotation. *A priori* testing of the explicit ARSM was done using DNS data of Okamoto & Shima (2005). Result obtained from (3.14) that is independent of the rotating frame (○) and results obtained from (3.15) for $\Omega_z = 0$ (—), for $\Omega_z = 1.1$ (– – –), and for $\Omega_z = 5$ (· · ·) are compared to the exact value of $a_{r\theta}$ (+).

in a rotating frame can be transformed to another rotating frame with the same accuracy. In this sense, the first frame-invariant model is appropriate as a general model.

4. Weak-equilibrium condition extended to curved and rotating flows

In the preceding section, we showed that a frame-invariant expression is desirable as a general model for the Reynolds stress. However, Gatski & Wallin (2004) argued that the weak-equilibrium condition $\mathrm{D}a^{\dagger}_{ij}/\mathrm{D}t = 0$ extended to curved and rotating flows is not objective and the resulting model is not frame-invariant. In this section, we will show that the condition $\mathrm{D}a^{\dagger}_{ij}/\mathrm{D}t = 0$ is in reality objective and hence the resulting model can be frame-invariant.

First, we describe the extended weak-equilibrium condition in an inertial frame (x^*_i system). The extension was proposed to take into account the effects of streamline curvature on turbulence (Girimaji 1997; Gatski & Jongen 2000). The condition $\mathrm{D}a^*_{ij}/\mathrm{D}t = 0$ exactly holds for stationary parallel mean flows. However, it is known that this condition is not suitable for curved flows. We then consider a locally defined rotating frame given by

$$x^{\dagger}_i = T_{ij}(x^*_j - x^*_{0j}), \tag{4.1}$$

where T_{ij} is an orthogonal transformation matrix and x^*_{0j} is the position of the origin of the local frame. The $x^{\dagger}_i$ system is chosen so that $|\mathrm{D}a^{\dagger}_{ij}/\mathrm{D}t|$ is minimized; the specific method will be mentioned later. The material derivative in the inertial frame can then be written as

$$\frac{\mathrm{D}a^*_{ij}}{\mathrm{D}t} = T^T_{ik}\frac{\mathrm{D}a^{\dagger}_{km}}{\mathrm{D}t}T_{mj} - \left(a^*_{ik}\Omega^{(r)*}_{kj} - \Omega^{(r)*}_{ik}a^*_{kj}\right), \tag{4.2}$$

where

$$\Omega_{ij}^{(r)*} \equiv \frac{\mathrm{d}T_{ik}^{T}}{\mathrm{d}t} T_{kj} = T_{ik}^{T} \Omega_{km}^{(r)\dagger} T_{mj}, \tag{4.3}$$

is the rotation rate of the $x_i^\dagger$ system expressed in the inertial frame and $\Omega_{ij}^{(r)\dagger}(= T_{ik}\mathrm{d}T_{kj}^{T}/\mathrm{d}t)$ is the rotation rate of the $x_i^\dagger$ system expressed in the $x_i^\dagger$ system. Expecting $|\mathrm{D}a_{ij}^\dagger/\mathrm{D}t|$ to be small enough, we assume the condition $\mathrm{D}a_{ij}^\dagger/\mathrm{D}t = 0$, resulting in

$$\frac{\mathrm{D}a_{ij}^{*}}{\mathrm{D}t} = -\left(a_{ik}^{*}\Omega_{kj}^{(r)*} - \Omega_{ik}^{(r)*}a_{kj}^{*}\right). \tag{4.4}$$

Therefore, the solution of the ARSM for flows in the inertial frame is given in the form

$$a_{ij}^{*} = f_{ij}\left(\hat{S}_{km}^{*}, \hat{W}_{km}^{*} - \frac{\tau}{A_0}\Omega_{km}^{(r)*}\right). \tag{4.5}$$

Once an appropriate $x_i^\dagger$ system is found, the ARSM can be improved by replacing $\hat{W}_{km}^{*}$ by $\hat{W}_{km}^{*} - (\tau/A_0)\Omega_{km}^{(r)*}$ in (3.6).

The choice of the $x_i^\dagger$ system affects the adequacy of the model. There are a few proposals for finding an appropriate $x_i^\dagger$ system. Girimaji (1997) proposed an $x_i^\dagger$ system based on the direction of the acceleration $\mathrm{D}U_i/\mathrm{D}t$. Gatski & Jongen (2000) used an $x_i^\dagger$ system aligned with the principal axes of the mean strain-rate tensor S_{ij}. Wallin & Johansson (2002) examined the two methods in more detail for three-dimensional flows. In this paper, we do not discuss this further; it is assumed that an appropriate $x_i^\dagger$ system is chosen in some way.

Next, we describe the extended weak-equilibrium condition in a rotating frame (x_i system). The condition (4.4) for the inertial frame can be transformed to the following condition for the rotating frame:

$$\frac{\bar{\mathrm{D}}a_{ij}}{\mathrm{D}t} = -(a_{ik}\Omega_{kj}^{(r)} - \Omega_{ik}^{(r)}a_{kj}), \tag{4.6}$$

where

$$\Omega_{ij}^{(r)} = Q_{ik}\Omega_{km}^{(r)*}Q_{mj}^{T}, \tag{4.7}$$

is the rotation rate of the $x_i^\dagger$ system expressed in the x_i system. The resulting solution can be written

$$a_{ij} = f_{ij}\left(\hat{S}_{km}, \hat{\bar{W}}_{km} - \frac{\tau}{A_0}\Omega_{km}^{(r)}\right), \tag{4.8}$$

using the same function f_{ij} as in (3.6). The above expression is different from (3.9) because Ω_{ij} is replaced by $\Omega_{ij}^{(r)}$. In contrast to Ω_{ij} the rotation rate $\Omega_{ij}^{(r)}$ is objective because it transforms as (4.7). Therefore, the model expression (4.8) is frame-invariant since it is written in terms of objective tensors only.

It is important to note that $\Omega_{ij}^{(r)}$ is determined by specifying the $x_i^\dagger$ system independently of the frame of the observer. Hence the condition $\mathrm{D}a_{ij}^\dagger/\mathrm{D}t = 0$ and its expression (4.6) in the rotating frame are objective. For a rotating flow in which the $x_i^\dagger$ system is trivial such as a rotating channel flow, $\Omega_{ij}^{(r)} = \Omega_{ij}$ is obtained for the observer in the trivial $x_i^\dagger$ system. In this case the expression (4.8) coincides with the expression (3.9). In general cases the expression (4.8) should be used. The condition (4.6) and the resulting expression (4.8) are objective; that is, a model making use of the extended weak-equilibrium condition can be frame-invariant. The adequacy of the model is of course coupled to the choice of the $x_i^{\dagger}$ system, but not to the choice of the system of the observer. Simulations of curved flows are usually performed in the inertial frame. Equation (4.8) means that such simulations can be done in any rotating frame including the inertial frame if the rotation rate $\Omega_{ij}^{(r)}$ is determined in an objective manner.

We have discussed Euclidean invariance of a model expression for the Reynolds stress because it is an important issue not only for the ARSM but also for general nonlinear eddy-viscosity models. The frame-invariance can be a useful constraint for theoretical modelling of nonlinear eddy-viscosity models such as the two-scale direct interaction approximation (Yoshizawa 1984, 1998). A model expression for the Reynolds stress in an inertial frame that contains the mean vorticity can be extended to a rotating frame simply by replacing the mean vorticity by the mean absolute vorticity (Yoshizawa 1998; Nagano & Hattori 2002). The frame-invariance does not contradict the extended weak-equilibrium condition described by Gatski & Wallin (2004) and hence the solution of the ARSM extended to curved and rotating flows can also be expressed in a frame-invariant way.

5. Conclusions

Since the corotational derivative of the Reynolds stress is objective under Euclidean transformations like the mean absolute vorticity tensor, the transport equation for the Reynolds stress can be written in a frame-invariant form. It was shown that a frame-invariant expression is desirable as a general model for the Reynolds stress by comparing the error of model expressions in different rotating frames. As pointed out by Gatski & Wallin (2004), the objective weak-equilibrium condition proposed by Weis & Hutter (2003) is not correct in actual rotating flows such as a rotating channel flow. However, this fact does not necessarily mean that weak-equilibrium conditions do not have to be objective. It was shown that the extended weak-equilibrium condition described by Gatski & Wallin (2004) is in reality objective and this condition does not contradict a frame-invariant model expression for the Reynolds stress. The frame-invariance can be a useful constraint for theoretical modelling of nonlinear eddy-viscosity models.

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